EXERCISE – III HINTS & SOLUTIONS

Sol.1 $y^2 = 4ax$ L.R. extreme

$P(4a, 4a)$ $L_1(a, 2a)$

Let $P(at_1^2, 2at_1)$ $L_1(at_2^2, 2at_2)$

$2at_1 = 4a$ $2at_2 = 2a$

$t_1 = 2$ $t_2 = 1$

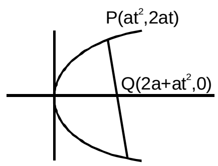
POI of two normals

$[a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2)]$

$[9a, -6a]$

which satisfy $y^2 = 4ax$ parabola

Sol.2 Equation of normal



$y + tx = 2at + at^3$

put $y = 0$

$Q(2a + at^2, 0)$

let mid point of PQ = M(h, k)

$h = \dfrac{at^2 + 2a + at^2}{2}$ $k = \dfrac{2at}{2}$

$t = \dfrac{k}{a}$

$h = at^2 + a$ ....(i)

$h = a.\dfrac{k^2}{a^2} + a$

$k^2 = a(h - a)$

$y^2 = a(x - a)$ this is a parabola

vertex $(a, 0)$

L.R. $= 4\left(\dfrac{a}{4}\right) = a$

Sol.3 $y^2 = 16x$ $4a = 16 \Rightarrow a = 4$

Let the POC $(at^2, 2at)$

Tangent

$ty = x + at^2$

slope $= \dfrac{1}{t} = 2$ and $\dfrac{1}{t} = \dfrac{-1}{2}$

$t = \dfrac{1}{2}$ $t = -2$

POC (1, 4) POC (16, –16)

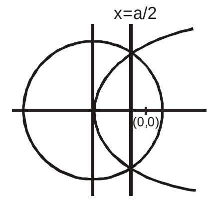
equation $\dfrac{y}{2} = x + \dfrac{a}{4}$ equation

$y = 2x + 2$ $-2y = x + 4 \times 4$

$2y + x + 16 = 0$

Sol.4 $x^2 + y^2 = \left(\dfrac{3a}{2}\right)^2$ sin a = 3a

$y^2 = 4ax$



$x^2 + 4ax - \dfrac{9a^2}{4} = 0$

$x = \dfrac{a}{2}, -\dfrac{3a}{2}$ (Reject)

Sol.5 $y^2 = 12x$ $4a = 12 \Rightarrow a = 3$

POI of tangent $[3t_1t_2, 3(t_1 + t_2)]$

$3t_1t_2 = 2$ ......(1)

$3(t_1 + t_2) = 5$ ......(2)

$t_1 - t_2 = \sqrt{(t_1 + t_2)^2 - 4t_1t_2} = \sqrt{\dfrac{25}{9} - \dfrac{8}{3}}$

$t_1 - t_2 = \dfrac{1}{3}$ ......(3)

By (2) and (3)

$t_1 = 1, \quad t_2 = \dfrac{2}{3}$

Tangent

$ty = x + at^2$ and

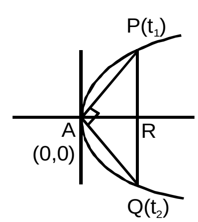
$ty = x + 3t^2$ $\dfrac{2}{3}y = x + 3 \times \dfrac{4}{9}$

$y = x + 3$ $2y = 3x + 4$

Sol.6 $P(at_1^2, 2at_1)$ $a = 1$

$P(t_1^2, 2t_1)$

$Q(t_2^2, 2t_2)$



$m_{AP} = \dfrac{2}{t_1}$ $m_{AQ} = \dfrac{2}{t_2}$

$\dfrac{2}{t_1} \times \dfrac{2}{t_2} = -4 \Rightarrow t_1t_2 = -4$

Fixed point (4, 0)

Let the middle point M(h, k)

$h = \dfrac{t_1^2 + t_2^2}{2} \Rightarrow t_1^2 + t_2^2 = 2h$ ......(1)

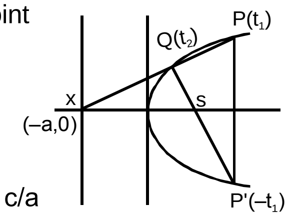
$k = \dfrac{2t_1 + 2t_2}{2} \Rightarrow t_1 + t_2 = k$ ......(2)

From (1) $(t_1 + t_2)^2 - 2t_1t_2 = 2h$

$k^2 + 8 = 2h$

$k^2 = 2(h - 4) \Rightarrow y^2 = 2(x - 4)$

Sol.7 If a chord passes through a fixed point on x-axis than



$t_1t_2 = -c/a$

$c = -a$

$t_1t_2 = 1$ .....(1)

$(-t_1)(t_2) = -1 = -c/a$

$-\dfrac{c}{a} = -1 \Rightarrow c = a$

fixed point (a, 0)

Sol.8 $y^2 = 4x$

normal equation in slope form

$y = mx - 2am - am^3$; $a = 1$

$y = mx - 2m - m^3$

passes through (15, 12)

$12 = 15m - 2m - m^3$

$m^3 - 13m + 12 = 0$

$\Rightarrow m = 1, m = -4, m = 3$

normals are

$m = 1 \Rightarrow y = x - 3$

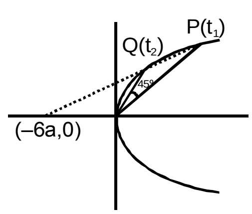
$m = -4 \Rightarrow y = -4x + 72$

$m = 3 \Rightarrow y = 3x - 33$

Sol.9 $y^2 = 4ax$

$t_1t_2 = 6$ .....(i)

$\tan 45^\circ = \left|\dfrac{\dfrac{2}{t_1} - \dfrac{2}{t_2}}{1 + \dfrac{4}{t_1t_2}}\right|$



$= 2\left|\dfrac{t_2 - t_1}{4 + t_1t_2}\right| = \left|\dfrac{t_2 - t_1}{5}\right|$

$(t_2 - t_1)^2 = 25$

$(t_1 + t_2)^2 - 4t_1t_2 = 25$

$t_1 + t_2 = \pm 7$

equation of chord

$2x - (t_1 + t_2)y + 2at_1t_2 = 0$

$(t_1 + t_2)y = 2x + 2at_1t_2$

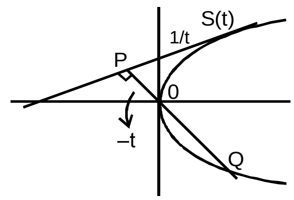
$\pm 7y = 2x + 12a$

Sol.10 We wish to prove that

$OP \cdot OQ$ = const.

tangent at A

$ty = x + at^2$



$OP = \dfrac{at^2}{\sqrt{1+t^2}}$

equation of OQ

$y = -tx$

$y^2 = 4ax$

$t^2x^2 = 4ax$

$x = \dfrac{4a}{t^2}$; $y = -\dfrac{4a}{t}$

$Q\left(\dfrac{4a}{t^2}, -\dfrac{4a}{t}\right)$

$OQ = \dfrac{4a}{t^2}\sqrt{1+t^2}$

$OP \cdot OQ = \dfrac{at^2}{\sqrt{1+t^2}} \times \dfrac{4a}{t^2}\sqrt{1+t^2}$

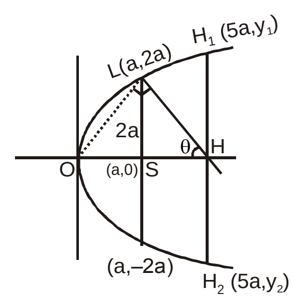
$= 4a^2$ constant

Sol.11 $M_{OL} = 2$

so $M_{LH} = -1/2$

$\tan\theta = 1/2$

$\dfrac{LS}{SH} = \dfrac{1}{2}$



$SH = 2 \times LS$

$= 2 \times 2a$

$SH = 4a$

so $H(4a + a, 0)$

$H(5a, 0)$

$5a = at^2$

$t = \pm\sqrt{5}$

$y_1 = 2at = \pm 2a\sqrt{5}$

length of double ordinate $= 2a\sqrt{5} + 2a\sqrt{5}$

$= 4a\sqrt{5}$

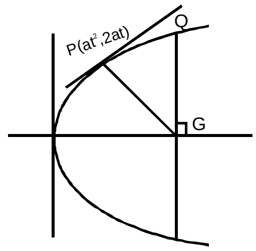
Sol.12 Equation of normal

$y + tx = 2at + at^3$

For G point

$y = 0$

$x = 2a + at^2$



$G\ (2a + at^2, 0)$

$R\ (2a + at^2, y)$

$PG^2 = 4a^2 + 4a^2t^2$

$QG^2 = y^2 = 4a\ (2a + at^2)$

$= 8a^2 + 4a^2t^2$

$QG^2 - PQ^2 = 8a^2 + 4a^2t^2 - 4a^2 - 4a^2t^2 = 4a^2$

Sol.13 $y^2 = 8x$; $a = 2$

$P(18, 12)$

$P(at_1^2, 2at_1)$

$P(2t_1^2, 4t_1)$

$2t_1^2 = 18$

$t_1 = \pm 3$

$t_1 = 3$ $\qquad$ $t_1 = -3$

$P(18, 12)$ $\qquad$ $t_2 = -t_1 - \dfrac{2}{t_1}$

itself $\qquad$ $t_2 = 3 + \dfrac{2}{3} = \dfrac{11}{3}$

$Q\left(\dfrac{242}{9}, \dfrac{44}{3}\right)$ $\quad$ $P(18, 12)$

$PQ = \dfrac{80}{9}\sqrt{10} \Rightarrow 9PQ = 80\sqrt{10}$

Sol.14 $y^2 = 12x$ $\qquad$ $a = 3$

$(3, 6)$

$2at_1 = 6$

$t_1 = 1$

$t_2 = -t_1 - \dfrac{2}{t_1} = -3$

$(at_2^2, 2at_2) = (27, -18)$

equation of circle

$(x - 3)(x - 27) + (y - 6)(y + 18) = 0$

$x^2 + y^2 - 30x + 12y - 27 = 0$

Sol.15 $x^2 = 4y$

$2x = 4y' \Rightarrow y' = \left.\dfrac{x}{2}\right|_{(6,9)} = 3$

$y - 9 = 3(x - 6)$

$y - 3x + 9 = 0$

touches the above line at (6, 9) than equation of circle

$(x - 6)^2 + (y - a)^2 + \lambda(y - 3x + 9) = 0$

passes through focus (0, 1)

$\lambda = -10$

$(x - 6)^2 + (y - a)^2 - 10(y - 3x + 9) = 0$

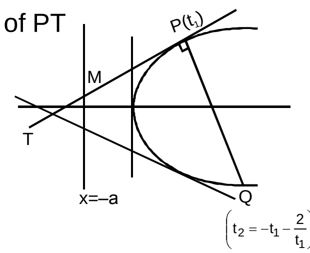
$x^2 + y^2 + 18x - 28y + 27 = 0$

Sol.16 $t_2 = -t_1 - \dfrac{2}{t_1}$

$\Rightarrow t_2 + t_1 = -\dfrac{2}{t_1}$ ....(1)

M is the mid point of PT

M(h, k)



$h = \dfrac{at_1^2 + at_1t_2}{2}$

$= \dfrac{at_1(t_1 + t_2)}{2}$

$= \dfrac{at_1}{2}\left(\dfrac{-2}{t_1}\right)$ $\qquad$ $T(at_1t_2, a(t_1 + t_2))$

$h = -a$

$x = -a$

Sol.17 $y^2 = 4x$

C.O.C. $T = 0$

$yy_1 = 2a(x + x_1)$

$2y = 2(x - 1)$

$y = x - 1$

area $= \dfrac{(y_1^2 - 4x_1)^{3/2}}{2a} = \dfrac{(4+4)^{3/2}}{2} = 8\sqrt{2}$

Sol.18 $x^2 = 4by$

Normal equation

$y = mx + 2b + \dfrac{b}{m^2}$ $\qquad$ Here b = 2

(h, k)

$k = mh + 4 + \dfrac{2}{m^2}$ ......(1)

$hm^3 + (4 - k)m^2 + 2 = 0 \rightarrow m_1, m_2, m_3$

$$m_1 + m_2 + m_3 = \frac{4-k}{h}$$

$$m_1m_2m_3 = -\frac{2}{h}$$

Sol.19 $m_1m_2m_3 = -\frac{2}{h}$

$m_1m_2 = -1$

$m_3 = \frac{2}{h}$

put m_3 in equation (1)

$k = 2 + 4 + \frac{2}{4/h^2}$

$k = 6 + \frac{h^2}{2}$

$h^2 = 2(k-6)$

$x^2 = 2(x-6)$

LR = 4a = 2

Sol.20 $m_1m_2 = 1$

$m_3 = \frac{-2}{h}$

put in equation (1)

$k = -2 + 4 + \frac{2}{4/h^2}$

$k = 2 + \frac{h^2}{2}$

$h^2 = 2(k-2)$

$x^2 = 2(y-2)$

Directrix : Y = – a

$y - 2 = -\frac{1}{2}$

$y = 2 - \frac{1}{2} = \frac{3}{2}$

$2y - 3 = 0$

Sol.21 $C_1 : y^2 = 4ax$

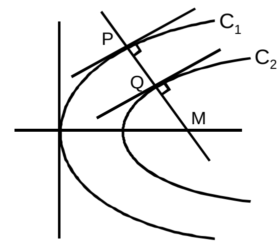
$C_2 : y^2 = c(x-b)$

let the slope of the common normal be 'm'

C_1 equation of normal $y = mx - 2am - am^3$ ....(1)

C_2 equation of normal $y = m(x-b) - 2cm - cm^3$

equation (1) & (2) represent same equation of straight line



$-2am - am^3 = -bm - 2cm - cm^3$

$m[2a + am^2 = b + 2c + cm^2]$

$m[2(a-c) - b + m^2(a-c)] = 0$

$m = 0$

$\Downarrow$

common normal is axis

$m^2(a-c) = b - 2(a-c)$

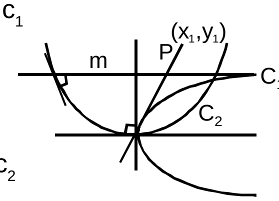
$m^2 = \frac{b}{a-c} - 2 > 0$

$\frac{b}{a-c} > 2$

Sol.22 Let the slope of common line is 'm'

$y = mx + \frac{a}{m}$

equation of tangent to c_1



$y = mx + 2b + \frac{b}{m^2}$

equation of normal at c_2

By comparison

$\frac{a}{m} = 2b + \frac{b}{m^2}$

$2bm^2 - am + b = 0$

D > 0

$a^2 - 8b^2 > 0$

$a^2 > 8b^2$